Problems of classifying associative or Lie algebras and triples of symmetric or skew-symmetric matrices are wild

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Abstract

We prove that the problems of classifying triples of symmetric or skew-symmetric matrices up to congruence, local commutative associative algebras with zero cube radical and square radical of dimension 3, and Lie algebras with central commutator subalgebra of dimension 3 are hopeless since each of them reduces to the problem of classifying pairs of n-by-n matrices up to simultaneous similarity.

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1 Introduction

All matrices, vector spaces, and algebras are considered over an algebraically closed field $\mathbb{F}$ of characteristic other than two.

The problem of classifying pairs of $n \times n$ matrices up to similarity transformations

$$(A, B) \longmapsto S^{-1}(A, B)S := (S^{-1}AS,\ S^{-1}BS),$$

in which S is any nonsingular $n \times n$ matrix, is hopeless since it contains the problem of classifying an arbitrary system of linear operators and the problem of classifying representations of an arbitrary finite-dimensional algebra, see [3]. Classification problems that contain the problem of classifying pairs of matrices up to similarity are called *wild*.

We prove the wildness of the problems of classifying

(i) triples of Hermitian forms (with respect to a nonidentity involution on $\mathbb{F}$),

(ii) for each $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$, triples of bilinear forms $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, in which $\mathcal{A}_i$ is symmetric if $\varepsilon_i = 1$ and skew-symmetric if $\varepsilon_i = -1$,

(iii) local commutative associative algebras Λ over $\mathbb{F}$ with $(\operatorname{Rad}\Lambda)^3 = 0$ and $\dim(\operatorname{Rad}\Lambda)^2 = 3$, and

(iv) Lie algebras L over $\mathbb{F}$ with central commutator subalgebra of dimension 3.

The hopelessness of the problems of classifying triples (i) and (ii) was also proved in [11] by another method (which was used in [12] too): each of them reduces to the problem of classifying representations of a wild quiver. The wildness of the problem of classifying local associative algebras Λ with $(\operatorname{Rad}\Lambda)^3 = 0$ and $\dim(\operatorname{Rad}\Lambda)^2 = 2$ was proved in [2].

Recall that an *algebra* Λ over $\mathbb{F}$ is a finite dimensional vector space being also a ring such that

$$\alpha(ab) = (\alpha a)b = a(\alpha b)$$

for all $\alpha \in \mathbb{F}$ and all $a, b \in \Lambda$. An algebra Λ is *local* if there exists an ideal R such that Λ/R is isomorphic to $\mathbb{F}$ (then R is the *radical* of Λ and is denoted by $\operatorname{Rad}\Lambda$).

A *Lie algebra L with central commutator subalgebra* is a vector space with multiplication given by a skew-symmetric bilinear mapping

$$[\ ,\ ]\colon L \times L \longrightarrow L$$

that satisfies $[[a, b], c] = 0$ for all $a, b, c \in L$. The *commutator subalgebra* L^2 is the subspace spanned by all $[a, b]$.

2 Triples of forms

Let $a \mapsto \bar{a}$ be any involution on $\mathbb{F}$, that is, a bijection $\mathbb{F} \to \mathbb{F}$ such that

$$\overline{a+b} = \bar{a} + \bar{b}, \quad \overline{ab} = \bar{a}\bar{b}, \quad \bar{\bar{a}} = a.$$

For a matrix $A = [a_{ij}]$, we define

$$A^* := \bar{A}^T = [\bar{a}_{ji}].$$

If $S^*AS = B$ for a nonsingular matrix S, then A and B are said to be **congruent*. The involution $a \mapsto \bar{a}$ can be the identity; we consider congruence of matrices as a special case of *congruence.

Each matrix tuple in this paper is formed by matrices of the same size, which is called the size of the tuple. Denote

$$R(A_1, \dots, A_t) := (RA_1, \dots, RA_t), \qquad (A_1, \dots, A_t)S := (A_1S, \dots, A_tS).$$

We say that matrix tuples $(A_1, \dots, A_t)$ and $(B_1, \dots, B_t)$ are *equivalent* and write

$$(A_1, \dots, A_t) \sim (B_1, \dots, B_t) \tag{1}$$

if there exist nonsingular R and S such that

$$R(A_1, \dots, A_t)S = (B_1, \dots, B_t).$$

These tuples are **congruent* if $R = S^*$.

Denote by I_n the $n \times n$ identity matrix, by 0_{mn} the $m \times n$ zero matrix, and abbreviate 0_{nn} to 0_n.

For $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{F}$, define the triple

$$\mathcal{T}_\varepsilon(x, y) := \left(\begin{bmatrix} 0_4 & I_4 \\ \varepsilon_1 I_4 & 0_4 \end{bmatrix}, \begin{bmatrix} 0_4 & J_4(0) \\ \varepsilon_2 J_4(0)^T & 0_4 \end{bmatrix}, \begin{bmatrix} 0_4 & D(x, y) \\ \varepsilon_3 D(x^*, y^*) & 0_4 \end{bmatrix} \right) \tag{2}$$

of polynomial matrices in $x,\ y,\ x^*$, and y^*, in which

$$J_4(0) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D(x,y) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

For each pair (A,B) of n-by-n matrices, define $\mathcal{T}_\varepsilon(A,B) =$

$$\left(\begin{bmatrix} 0_{4n} & I_{4n} \\ \varepsilon_1 I_{4n} & 0_{4n} \end{bmatrix}, \begin{bmatrix} 0_{4n} & J_4(0_n) \\ \varepsilon_2 J_4(0_n)^T & 0_{4n} \end{bmatrix}, \begin{bmatrix} 0_{4n} & D(A,B) \\ \varepsilon_3 D(A^*,B^*) & 0_{4n} \end{bmatrix} \right), \tag{4}$$

where

$$J_4(0_n) = \begin{bmatrix} 0_n & I_n & 0 & 0 \\ 0 & 0_n & I_n & 0 \\ 0 & 0 & 0_n & I_n \\ 0 & 0 & 0 & 0_n \end{bmatrix}, \qquad D(A,B) = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0_n \end{bmatrix}. \tag{5}$$

We prove in this section the following theorem; its statement (a) is used in the next section.

Theorem 1. *Let $\mathbb{F}$ be an algebraically closed field of characteristic other than two.*

(a) *For nonzero $\varepsilon_1, \varepsilon_2 \in \mathbb{F}$ and any $\varepsilon_3 \in \mathbb{F}$, matrix pairs (A,B) and (C,D) over $\mathbb{F}$ are similar if and only if $\mathcal{T}_\varepsilon(A,B)$ and $\mathcal{T}_\varepsilon(C,D)$ are *congruent.*

(b) *The problems of classifying triples* (i) *and* (ii) *from Section 1 are wild.*

Define the *direct sum* of matrix tuples:

$$(A_1,\dots,A_t) \oplus (B_1,\dots,B_t) := (A_1 \oplus B_1,\dots,A_t \oplus B_t).$$

We say that a tuple $\mathcal{T}_1$ of $p \times q$ matrices is a *direct summand of a tuple $\mathcal{T}$ for equivalence* if $p+q>0$ and $\mathcal{T}$ is equivalent to $\mathcal{T}_1 \oplus \mathcal{T}_2$ for some $\mathcal{T}_2$. If also $p=q$ and $\mathcal{T}$ is *congruent to $\mathcal{T}_1 \oplus \mathcal{T}_2$, then $\mathcal{T}_1$ is a *direct summand of $\mathcal{T}$ for *congruence.* A matrix tuple is *indecomposable* with respect to equivalence (*congruence) if it has no direct summand of a smaller size for equivalence (*congruence).

Lemma 2. (a) *Each tuple of m-by-n matrices is equivalent to a direct sum of tuples that are indecomposable with respect to equivalence. This sum is determined uniquely up to permutation of summands and replacement of summands by equivalent tuples.*

(b) *Each tuple of n-by-n matrices is *congruent to a direct sum of tuples that are indecomposable with respect to *congruence. This sum is determined uniquely up to permutation of summands and replacement of summands by *congruent tuples.*

Proof. (a) Each t-tuple of $m \times n$ matrices determines the t-tuple of linear mappings $\mathbb{F}^n \to \mathbb{F}^m$; that is, the representation of the quiver consisting of two vertices 1 and 2 and t arrows $1 \longrightarrow 2$. By the Krull–Schmidt theorem [7, Section 8.2], every representation of a quiver is isomorphic to a direct sum of indecomposable representations determined up to replacement by isomorphic representations and permutations of summands.

(b) This statement is a special case of the following generalization of the law of inertia for quadratic forms [12, Theorem 2 and § 2]: each system of linear mappings and sesquilinear forms on vector spaces over $\mathbb{F}$ decomposes into a direct sum of indecomposable systems uniquely up to isomorphisms of summands. □

It is worthy of note that the uniqueness of decompositions in Lemma 2(a) holds only if we suppose that there exists exactly one matrix of size $0 \times n$ and there exists exactly one matrix of size $n \times 0$ for every nonnegative integer n; they give the linear mappings $\mathbb{F}^n \to 0$ and $0 \to \mathbb{F}^n$ and are considered as zero matrices 0_{0n} and 0_{n0}. Then for any m-by-n matrix M

$$M \oplus 0_{p0} = \begin{bmatrix} M \\ 0_{pn} \end{bmatrix} \qquad \text{and} \qquad M \oplus 0_{0q} = \begin{bmatrix} M & 0_{mq} \end{bmatrix}. \tag{6}$$

In particular, $0_{p0} \oplus 0_{0q} = 0_{pq}$.

Lemma 3. (a) *Every direct summand for equivalence of*

$$\mathcal{G} = (I_{4n},\ J_4(0_n),\ D) \tag{7}$$

(*in which $J_4(0_n)$ is defined in* (5) *and D is any $4n$-by-$4n$ matrix*) *reduces by equivalence transformations to the form*

$$\mathcal{G}' = (I_{4p},\ J_4(0_p),\ M'). \tag{8}$$

(b) *If* (8) *is a direct summand for equivalence of the tuple* (7) *with*

$$D = \operatorname{diag}(\alpha I_n,\ A,\ B,\ \beta I_n) \tag{9}$$

(A and B are n-by-n and $\alpha, \beta \in \mathbb{F}$), then M' has the form

$$M' = \begin{bmatrix} \alpha I_p & M'_{12} & M'_{13} & M'_{14} \\ 0 & M'_{22} & M'_{23} & M'_{24} \\ 0 & 0 & M'_{33} & M'_{34} \\ 0 & 0 & 0 & \beta I_p \end{bmatrix} \tag{10}$$

(all blocks are p-by-p).

Proof. (a) Let $\mathcal{G}'$ be a direct summand for equivalence of the tuple (7); this means that $\mathcal{G} \sim \mathcal{G}' \oplus \mathcal{G}''$ (in the notation (1)) for some $\mathcal{G}''$. The first matrix of the triple $\mathcal{G}$ is the identity, so we can reduce the first matrix of $\mathcal{G}' \oplus \mathcal{G}''$ to the identity matrix too by equivalence transformations with $\mathcal{G}'$ and $\mathcal{G}''$. Then the equivalence of $\mathcal{G}$ and $\mathcal{G}' \oplus \mathcal{G}''$ means that their second matrices are similar. The second matrix of $\mathcal{G}$ is similar to the Jordan matrix $J_4(0) \oplus \cdots \oplus J_4(0)$, and so we can reduce the second matrix of $\mathcal{G}'$ to $J_4(0_p)$ by equivalence transformations with $\mathcal{G}'$. This proves (a).

(b) Let (8) be a direct summand for equivalence of the tuple (7) with D of the form (9). Then $\mathcal{G} \sim \mathcal{G}' \oplus \mathcal{G}''$ for some $\mathcal{G}''$. By (a), $\mathcal{G}''$ can be taken in the form $\mathcal{G}'' = (I_{4q},\ J_4(0_q),\ M'')$ with $q := n - p$. Partition M' and M'' into p-by-p and q-by-q blocks:

$$M' = [M'_{ij}]_{i,j=1}^4, \qquad M'' = [M''_{ij}]_{i,j=1}^4.$$

Using simultaneous permutations of rows and columns of the matrices of $\mathcal{G}' \oplus \mathcal{G}''$, we construct the equivalence

$$\mathcal{G}' \oplus \mathcal{G}'' \sim (I_{4n},\ J_4(0_n),\ M), \qquad M := [M'_{ij} \oplus M''_{ij}]_{i,j=1}^4. \tag{11}$$

Since $\mathcal{G} \sim \mathcal{G}' \oplus \mathcal{G}''$, we have $\mathcal{G} \sim (I_{4n},\ J_4(0_n),\ M)$, and so there exist nonsingular R and S such that

$$\mathcal{G}S = R(I_{4n},\ J_4(0_n),\ M). \tag{12}$$

Equating the corresponding matrices of the triples (12) gives

$$I_{4n}S = RI_{4n}, \qquad J_4(0_n)S = RJ_4(0_n), \qquad DS = RM.$$

By the first and the second equalities,

$$S = R = \begin{bmatrix} S_0 & S_1 & S_2 & S_3 \\ 0 & S_0 & S_1 & S_2 \\ 0 & 0 & S_0 & S_1 \\ 0 & 0 & 0 & S_0 \end{bmatrix}. \tag{13}$$

By the third equality and (9), M has the form

$$\begin{bmatrix} \alpha I_n & M_{12} & M_{13} & M_{14} \\ 0 & M_{22} & M_{23} & M_{24} \\ 0 & 0 & M_{33} & M_{34} \\ 0 & 0 & 0 & \beta I_n \end{bmatrix}.$$

Since M is defined by (11), M' has the form (10). $\square$

Proof of Theorem 1. (a) If (A, B) is similar to (C, D), then $\mathcal{T}_\varepsilon(A, B)$ is *congruent to $\mathcal{T}_\varepsilon(C, D)$ since $S^{-1}(A, B)S = (C, D)$ implies

$$R^*\mathcal{T}_\varepsilon(A, B)R = \mathcal{T}_\varepsilon(C, D),$$
$$R := \operatorname{diag}((S^*)^{-1}, (S^*)^{-1}, (S^*)^{-1}, (S^*)^{-1}, S, S, S, S).$$

Conversely, suppose that $\mathcal{T}_\varepsilon(A, B)$ is *congruent to $\mathcal{T}_\varepsilon(C, D)$. Then they are equivalent, and so

$$\mathcal{G}(A, B) \oplus \mathcal{H}_\varepsilon(A, B) \sim \mathcal{G}(C, D) \oplus \mathcal{H}_\varepsilon(C, D), \tag{14}$$

where

$$\begin{aligned} \mathcal{G}(X, Y) &:= (I_{4n},\ J_4(0_n),\ D(X, Y)), \\ \mathcal{H}_\varepsilon(X, Y) &:= (\varepsilon_1 I_{4n},\ \varepsilon_2 J_4(0_n)^T,\ \varepsilon_3 D(X^*, Y^*)) \end{aligned} \tag{15}$$

for n-by-n matrices X and Y. Let $\mu_2 := \varepsilon_2/\varepsilon_1$ and $\mu_3 := \varepsilon_3/\varepsilon_1$, then

$$\mathcal{H}_\varepsilon(X, Y) \sim \mathcal{H}_\mu(X, Y) = (I_{4n},\ \mu_2 J_4(0_n)^T,\ \mu_3 D(X^*, Y^*)).$$

Furthermore, let

$$S := \operatorname{diag}(I_n,\ \mu_2 I_n,\ \mu_2^2 I_n,\ \mu_2^3 I_n), \qquad \nu := \mu_3,$$

then

$$\mathcal{H}_\mu(X, Y) \sim S^{-1}\mathcal{H}_\mu(X, Y)S = (I_{4n},\ J_4(0_n)^T,\ \nu D(X^*, Y^*)) = \mathcal{H}_\nu(X, Y).$$

Lastly,

$$\mathcal{H}_\nu(X, Y) \sim P\mathcal{H}_\nu(X, Y)P = (I_{4n},\ J_4(0_n),\ \nu D'(X^*, Y^*)),$$

where

$$P = \begin{bmatrix} 0 & 0 & 0 & I_n \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix}, \qquad D'(X^*, Y^*) = \begin{bmatrix} 0_n & 0 & 0 & 0 \\ 0 & Y^* & 0 & 0 \\ 0 & 0 & X^* & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}.$$

Therefore,

$$\mathcal{H}_\varepsilon(X,Y) \sim \mathcal{H}'(X,Y) := (I_{4n},\ J_4(0_n),\ \nu D'(X^*, Y^*)),$$

and by (14)

$$\mathcal{G}(A,B) \oplus \mathcal{H}'(A,B) \sim \mathcal{G}(C,D) \oplus \mathcal{H}'(C,D). \tag{16}$$

Suppose that $\mathcal{G}(A,B)$ and $\mathcal{H}'(C,D)$ have a common direct summand $\mathcal{G}'$ for equivalence. By Lemma 3(a), we may take $\mathcal{G}' = (I_{4p},\ J_4(0_p),\ M')$. Moreover, since $D(A,B)$ and $\nu D'(C^*, D^*)$ are of the form (9) with $\alpha = 1$ and $\alpha = 0$, respectively, by Lemma 3(b) the matrix M' has the form (10) with $\alpha = 1$ and $\alpha = 0$ simultaneously, a contradiction.

Hence $\mathcal{G}(A,B)$ and $\mathcal{H}'(C,D)$ have no common direct summands for equivalence. The triples $\mathcal{G}(C,D)$ and $\mathcal{H}'(A,B)$ have no common direct summands too. By (16) and Lemma 2(a), $\mathcal{G}(A,B) \sim \mathcal{G}(C,D)$; that is, $\mathcal{G}(A,B)S = R\mathcal{G}(C,D)$ for some nonsingular R and S. Equating the corresponding matrices of these triples gives (13) and $(A,B)S_0 = S_0(C,D)$; that is, (A,B) is similar to (C,D).

(b) If the involution on $\mathbb{F}$ is not the identity and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, then the matrices of the triple (4) are Hermitian. If the involution on $\mathbb{F}$ is the identity and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$, then each matrix of the triple (4) is symmetric or skew-symmetric. So the statement (b) of Theorem 1 follows from the statement (a). □

3 Algebras

We consider (associative) algebras and Lie algebras as special cases of semialgebras. By a *semialgebra* we mean a finite-dimensional vector space R over $\mathbb{F}$ with multiplication given by a bilinear mapping $(a,b) \mapsto ab \in R$:

$$(\alpha a + \beta b)c = \alpha(ac) + \beta(bc), \qquad a(\alpha b + \beta c) = \alpha(ab) + \beta(ac)$$

for all $\alpha, \beta \in \mathbb{F}$ and all $a, b, c \in R$. A semialgebra R is *commutative* or *anti-commutative* if $ab = ba$ or, respectively, $ab = -ba$ for all $a, b \in R$. Denote by R^2 and R^3 the vector spaces spanned by all ab and, respectively, by all $(ab)c$ and $a(bc)$, where $a, b, c \in R$.

An *algebra* Λ over $\mathbb{F}$ is an associative semialgebra with the identity 1:

$$(ab)c = a(bc), \qquad 1a = a \qquad (a, b, c \in \Lambda).$$

An algebra Λ is *local* if the set R of its noninvertible elements is closed under addition. Then R is the *radical* and Λ/R is isomorphic to $\mathbb{F}$ (see [7, Section 5.2]).

A *Lie algebra* L over $\mathbb{F}$ is an anti-commutative semialgebra whose multiplication is denoted by [,] and satisfies the Jacobi identity

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0 \tag{17}$$

for all $a, b, c \in L$. Then L^2 is called the *commutator subalgebra* of L. The commutator subalgebra is *central* if $L^3 = 0$, that is, if

$$[[a, b], c] = 0 \quad \text{for all } a, b, c \in L;$$

the last equality implies (17). A Lie algebra with central commutator subalgebra is also called a *two-step nilpotent Lie algebra*. Due to the next theorem, the full classification of such Lie algebras is impossible; one can consider its special cases or reduce it to another classification problem of the same complexity; see, for instance, [6, Theorems 2 and 3].

Theorem 4. *Let $\mathbb{F}$ be an algebraically closed field of characteristic other than two.*

(a) *The problem of classifying local commutative algebras Λ over $\mathbb{F}$ with $(\operatorname{Rad}\Lambda)^3 = 0$ and $\dim(\operatorname{Rad}\Lambda)^2 = 3$ is wild.*

(b) *The problem of classifying Lie algebras over $\mathbb{F}$ with central commutator subalgebra of dimension 3 is wild.*

By the next lemma, the problems considered in Theorem 4 are the problems of classifying commutative or anti-commutative semialgebras R with $R^3 = 0$ and $\dim R^2 = 3$ (these semialgebras are associative and satisfy (17) due to $R^3 = 0$).

Lemma 5. *Let R be a semialgebra with $R^3 = 0$ and $\dim R^2 = 3$.*

(a) *R is commutative if and only if R is the radical of some algebra Λ from Theorem* 4(a)*; moreover, Λ is fully determined by R.*

(b) *R is anti-commutative if and only if R is a Lie algebra from Theorem* 4(b)*.*

Proof. Let R be a semialgebra with $R^3 = 0$ and $\dim R^2 = 3$.

(a) If R is commutative, then we "adjoin" the identity 1 by considering the algebra Λ consisting of the formal sums

$$\alpha 1 + a \qquad (\alpha \in \mathbb{F},\ a \in R)$$

with the componentwise addition and scalar multiplication and the multiplication

$$(\alpha 1 + a)(\beta 1 + b) = \alpha\beta 1 + (\alpha b + \beta a + ab).$$

This multiplication is associative since $R^3 = 0$, and so Λ is a commutative algebra. Since R is the set of its noninvertible elements, Λ is a local algebra and R is its radical.

(b) If R is anti-commutative, then R is a Lie algebra since (17) holds due to $R^3 = 0$. $\square$

Lemma 6. *Every semialgebra R with $R^3 = 0$ and $\dim R^2 = t$ is isomorphic to exactly one semialgebra on $\mathbb{F}^n$ with multiplication*

$$uv = \left(u^T \begin{bmatrix} 0_t & 0 \\ 0 & A_1 \end{bmatrix} v, \ldots, u^T \begin{bmatrix} 0_t & 0 \\ 0 & A_t \end{bmatrix} v,\, 0, \ldots, 0\right)^T, \tag{18}$$

given by a tuple $(A_1, \ldots, A_t)$ of $(n-t)$-by-$(n-t)$ matrices that are linearly independent; this means that for all $\alpha_1, \ldots, \alpha_t \in \mathbb{F}$

$$\alpha_1 A_1 + \cdots + \alpha_t A_t = 0 \qquad \Longrightarrow \qquad \alpha_1 = \cdots = \alpha_t = 0.$$

The tuple $(A_1, \ldots, A_t)$ is determined by R uniquely up to congruence and linear substitutions

$$(A_1, \ldots, A_t) \longmapsto (\gamma_{11} A_1 + \cdots + \gamma_{1t} A_t,\ \ldots,\ \gamma_{t1} A_1 + \cdots + \gamma_{tt} A_t), \tag{19}$$

in which the matrix $[\gamma_{ij}]$ must be nonsingular. The semialgebra R is commutative or anti-commutative if and only if all the matrices $A_1, \ldots, A_t$ are symmetric or, respectively, skew-symmetric.

Proof. Let R be a semialgebra of dimension n with $R^3 = 0$ and $\dim R^2 = t$. Choose a basis $e_1, \dots, e_t$ of R^2 and complete it to a basis

$$e_1, \dots, e_t, f_1, \dots, f_{n-t} \tag{20}$$

of R. Since $R^3 = 0$,

$$e_i e_j = 0, \quad e_i f_j = 0, \quad f_i f_j = \alpha_{1ij} e_1 + \dots + \alpha_{tij} e_t, \tag{21}$$

and the $(n-t)$-by-$(n-t)$ matrices $A_1 = [\alpha_{1ij}], \dots, A_t = [\alpha_{tij}]$ are symmetric or skew-symmetric if R is commutative or, respectively, anti-commutative. Representing the elements of R by their coordinate vectors with respect to the basis (20) and using (21), we obtain (18). A change of the basis $e_1, \dots, e_t$ of R^2 reduces $(A_1, \dots, A_t)$ by transformations (19). A change of the basis vectors $f_1, \dots, f_{n-t}$ reduces $(A_1, \dots, A_t)$ by congruence transformations. The linear independence of the system of matrices $A_1, \dots, A_t$ follows from (21) because $\dim R^2 = t$. □

Due to Lemma 6 and the next lemma, the problem of classifying commutative (respectively, anti-commutative) semialgebras R with $R^3 = 0$ and $\dim R^2 = 3$ is wild. By Lemma 5, *this proves Theorem* 4.

Lemma 7. *The problem of classifying triples of symmetric* (*respectively, skew-symmetric*) *matrices up to congruence and substitutions* (19) *with* $t = 3$ *is wild.*

Proof. Let $\varepsilon = 1$ (respectively, $\varepsilon = -1$), denote

$$A^{\triangledown} = \begin{bmatrix} 0 & A \\ \varepsilon A^T & 0 \end{bmatrix}$$

for each matrix A, and denote

$$(A, \dots, D)^{\triangledown} = (A^{\triangledown}, \dots, D^{\triangledown}) \tag{22}$$

for each matrix tuple $(A, \dots, D)$.

Consider the triple of 350-by-350 matrices

$$\mathcal{T}(x, y) := (I_{100}, 0_{100}, 0_{100})^{\triangledown} \oplus (0_{50}, I_{50}, 0_{50})^{\triangledown} \oplus (0_{20}, 0_{20}, I_{20})^{\triangledown} \oplus (I_1, I_1, I_1)^{\triangledown} \oplus \mathcal{G}(x, y)^{\triangledown}, \tag{23}$$

in which
$$\mathcal{G}(x,y)=(I_4,\ J_4(0),\ D(x,y)) \tag{24}$$
(see (3) and (15)).

Let (A,B) and (C,D) be two pairs of n-by-n matrices. If (A,B) is similar to (C,D); that is, $S^{-1}(A,B)S=(C,D)$ for some nonsingular S, then $\mathcal{G}(A,B)^{\triangledown}$ is congruent to $\mathcal{G}(C,D)^{\triangledown}$ since

$$R^T\mathcal{G}(A,B)^{\triangledown}R=\mathcal{G}(C,D)^{\triangledown},$$

where

$$R:=(S^T)^{-1}\oplus(S^T)^{-1}\oplus(S^T)^{-1}\oplus(S^T)^{-1}\oplus S\oplus S\oplus S\oplus S.$$

Hence, $\mathcal{T}(A,B)$ is congruent to $\mathcal{T}(C,D)$.

Conversely, assume that $\mathcal{T}(A,B)$ reduces to $\mathcal{T}(C,D)$ by congruence transformations and substitutions (19); we need to prove that (A,B) is similar to (C,D). These transformations are independent; we can first produce all substitutions reducing

$$(M_1,M_2,M_3(A,B)):=\mathcal{T}(A,B)$$

to
$$(\gamma_{i1}M_1+\gamma_{i2}M_2+\gamma_{i3}M_3(A,B))_{i=1}^3 \qquad ([\gamma_{ij}]\text{ is nonsingular}), \tag{25}$$
and then all congruence transformations and obtain

$$(M_1,M_2,M_3(C,D))=\mathcal{T}(C,D). \tag{26}$$

Since (25) and (26) are congruent,

$$\operatorname{rank}(\gamma_{i1}M_1+\gamma_{i2}M_2+\gamma_{i3}M_3(A,B))=\begin{cases}\operatorname{rank}M_i & \text{if } i=1 \text{ or } i=2,\\ \operatorname{rank}M_3(C,D) & \text{if } i=3,\end{cases}$$

and so $\gamma_{ij}=0$ if $i\neq j$ because of the form (23) of matrices of $\mathcal{T}(x,y)$; that is, $\mathcal{T}(C,D)$ is congruent to

$$(\gamma_{11}M_1,\ \gamma_{22}M_2,\ \gamma_{33}M_3(A,B)).$$

Since $\mathbb{F}$ is algebraically closed, the last triple is congruent to

$$\gamma_{11}^{-1/2}(\gamma_{11}M_1,\ \gamma_{22}M_2,\ \gamma_{33}M_3(A,B))\gamma_{11}^{-1/2}.$$

Hence,

$$\mathcal{T}(C,D) \text{ is congruent to } (M_1,\, \alpha M_2,\, \beta M_3(A,B)),$$

in which $\alpha := \gamma_{22}/\gamma_{11}$ and $\beta := \gamma_{33}/\gamma_{11}$.

By (23), $(I_1,\, I_1,\, I_1)^{\triangledown}$ is a direct summand of $(M_1,\, M_2,\, M_3(A,B))$ for congruence. Hence, $(I_1,\, \alpha I_1,\, \beta I_1)^{\triangledown}$ is a direct summand of $(M_1,\, \alpha M_2,\, \beta M_3(A,B))$ for congruence. Lemma 2(b) ensures that each decomposition of $\mathcal{T}(C,D)$ by congruence transformations into a direct sum of indecomposable triples must have a direct summand that is congruent to $(I_1,\, \alpha I_1,\, \beta I_1)^{\triangledown}$.

By simultaneous permutations of rows and columns, $\mathcal{T}(C,D)$ reduces to a direct sum of triples of the form

$$(I_1, 0_1, 0_1)^{\triangledown}, \quad (0_1, I_1, 0_1)^{\triangledown}, \quad (0_1, 0_1, I_1)^{\triangledown}, \quad (I_1, I_1, I_1)^{\triangledown}, \tag{27}$$

and of the triple $\mathcal{G}(C,D)^{\triangledown}$ defined in (24).

The triple $\mathcal{G}(C,D)^{\triangledown}$ has no direct summand $(I_1,\, \alpha I_1,\, \beta I_1)^{\triangledown}$ for congruence since the pair obtained from $\mathcal{G}(C,D)^{\triangledown}$ by deleting its last matrix is permutationally congruent to

$$(I_4, J_4(0))^{\triangledown} \oplus \cdots \oplus (I_4, J_4(0))^{\triangledown} \quad (n \text{ summands});$$

this pair has no direct summand $(I_1, \alpha I_1)^{\triangledown}$ for equivalence, and so for congruence too. By Lemma 2(b), $(I_1,\, \alpha I_1,\, \beta I_1)^{\triangledown}$ is congruent to one of the triples (27), hence it is congruent to $(I_1, I_1, I_1)^{\triangledown}$, and so $\alpha = \beta = 1$; that is, $\mathcal{T}(A,B)$ is congruent to $\mathcal{T}(C,D)$. Due to (23), all the direct summands of $\mathcal{T}(A,B)$ and $\mathcal{T}(C,D)$ coincide except for $\mathcal{G}(A,B)^{\triangledown}$ and $\mathcal{G}(C,D)^{\triangledown}$. By Lemma 2(b), the triples $\mathcal{G}(A,B)^{\triangledown}$ and $\mathcal{G}(C,D)^{\triangledown}$ are congruent. By Theorem 1(a), (A,B) and (C,D) are similar. $\square$

Corollary 8. *Let U and V be vector spaces and $\dim V = 3$. The problem of classifying tensors $T \in U \otimes U \otimes V$ that are symmetric (respectively, skew-symmetric) on U is wild since it reduces to the classification problems considered in Lemma 7.*

4 Description of Lie algebras with central commutator subalgebra of dimension 2

In this section, we describe Lie algebras with central commutator subalgebra of dimension 2 using the canonical form of pairs of skew-symmetric matrices

for congruence. An analogous description of local commutative algebras with $(\operatorname{Rad}\Lambda)^3 = 0$ and $\dim(\operatorname{Rad}\Lambda)^2 = 3$ would be more awkward since the classification of pairs of symmetric matrices up to congruence is more complicated (see Thompson's article [13] with an extensive bibliography, or [12, Theorem 4]).

The problem of classifying Lie algebras with central commutator subalgebra of dimension 1 is trivial: by Lemma 6 each of them is isomorphic to exactly one algebra on $\mathbb{F}^n$ with multiplication

$$[u,v] := \left(u^T \begin{bmatrix} 0_p & 0 & 0 \\ 0 & 0 & I_q \\ 0 & -I_q & 0 \end{bmatrix} v,\, 0, \dots, 0 \right)^T,$$

given by natural numbers p and q such that $p + 2q = n$.

Define the $(m-1)$-by-m matrices

$$F_m = \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \quad G_m = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}$$

for each natural number m. In particular, $F_1 = G_1 = 0_{01}$ and so $(F_1, G_1)^{\triangledown} = (0_1, 0_1)$ by (6).

Theorem 9. *Let $\mathbb{F}$ be an algebraically closed field of characteristic other than two. Let L be a Lie algebra over $\mathbb{F}$ whose commutator subalgebra is central and has dimension 2. Then L is isomorphic to an algebra on $\mathbb{F}^n$ with multiplication*

$$[u,v] := \left(u^T \begin{bmatrix} 0_2 & 0 \\ 0 & A \end{bmatrix} v,\, u^T \begin{bmatrix} 0_2 & 0 \\ 0 & B \end{bmatrix} v,\, 0, \dots, 0 \right)^T, \tag{28}$$

given by a pair (A, B) of skew-symmetric $(n-2)$-by-$(n-2)$ matrices of the form

$$\bigoplus_{i=1}^{p} (I_{l_i}, J_{l_i}(\lambda_i))^{\triangledown} \oplus \bigoplus_{j=1}^{q} (F_{r_j}, G_{r_j})^{\triangledown}, \qquad p \geqslant 0,\ q \geqslant 0, \tag{29}$$

($J_l(\lambda)$ denotes the l-by-l Jordan block with eigenvalue λ, and $(\dots)^{\triangledown}$ is defined in (22) with $\varepsilon = -1$) except for the case

$$\lambda_1 = \dots = \lambda_p, \qquad l_1 = \dots = l_p = r_1 = \dots = r_q = 1. \tag{30}$$

The sum (29) is determined by L uniquely up to permutations of summands and up to linear-fractional transformations of the sequence of eigenvalues

$$(\lambda_1, \dots, \lambda_p) \longmapsto \left(\frac{\gamma + \delta\lambda_1}{\alpha + \beta\lambda_1}, \dots, \frac{\gamma + \delta\lambda_p}{\alpha + \beta\lambda_p} \right), \tag{31}$$

in which all $\alpha + \beta\lambda_i$ are nonzero and $\alpha\delta - \beta\gamma \neq 0$.

Proof. By [8], [12], or [13], each pair of skew-symmetric matrices over $\mathbb{F}$ is congruent to a direct sum of pairs of the form

$$(I_m, J_m(\lambda))^{\triangledown}, \qquad (J_m(0), I_m)^{\triangledown}, \qquad (F_m, G_m)^{\triangledown} \tag{32}$$

(in the notation (22) with $\varepsilon = -1$), and this sum is determined uniquely up to permutation of summands.

Let L be a Lie algebra of dimension n whose commutator subalgebra is central and has dimension 2. By Lemma 6 for $t = 2$, L is isomorphic to an algebra on $\mathbb{F}^n$ with multiplication (28) given by a pair (A, B) of skew-symmetric $(n-2)$-by-$(n-2)$ matrices, and (A, B) is determined by L uniquely up to congruence and invertible linear substitutions

$$(A, B) \longmapsto (\alpha A + \beta B,\, \gamma A + \delta B), \qquad \alpha\delta - \beta\gamma \neq 0. \tag{33}$$

By (32), the pair (A, B) is congruent to a pair

$$\bigoplus_{i=1}^{k} (I_{l_i}, J_{l_i}(\lambda_i))^{\triangledown} \oplus \bigoplus_{i=k+1}^{p} (J_{l_i}(0), I_{l_i})^{\triangledown} \oplus \bigoplus_{j=1}^{q} (F_{r_j}, G_{r_j})^{\triangledown} \tag{34}$$

determined by (A, B) uniquely up to permutation of summands.

Let us study how transformations (33) change (34). The pairs (32) are indecomposable with respect to congruence. Since the first and the second pairs in (32) have size $2m \times 2m$, each indecomposable pair of skew-symmetric matrices of size $(2m+1) \times (2m+1)$ is congruent to $(F_m, G_m)^{\triangledown}$. Each transformation (33) is invertible, so it transforms any indecomposable pair of skew-symmetric matrices to an indecomposable one. Hence, although each transformation (33) with (34) may spoil summands $(F_{r_j}, G_{r_j})^{\triangledown}$, but they are restored by congruence transformations.

If $k < p$, then we reduce the pair (34) to a pair of the form (29) (with other $\lambda_1, \dots, \lambda_k$) as follows. We convert all the summands $(I_{l_i}, J_{l_i}(\lambda_i))^{\triangledown}$ and

$(J_{l_i}(0), I_{l_i})^\triangledown$ to pairs with nonsingular first matrices by any transformation (33) given by

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} = \begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}, \quad \beta \neq 0,\ 1+\beta\lambda_1 \neq 0,\ \dots,\ 1+\beta\lambda_k \neq 0.$$

Then we reduce each of these summands to $(I, J_{l_i}(\lambda_i))^\triangledown$ (with other λ_i's) by congruence transformations using the following fact: if matrix pairs (M_1, M_2) and (N_1, N_2) are equivalent, i.e., $R(M_1, M_2)S = (N_1, N_2)$ for some nonsingular R and S, then $(M_1, M_2)^\triangledown$ and $(N_1, N_2)^\triangledown$ are congruent:

$$\begin{bmatrix}R & 0\\ 0 & S^T\end{bmatrix}\begin{bmatrix}0 & M_i\\ -M_i^T & 0\end{bmatrix}\begin{bmatrix}R^T & 0\\ 0 & S\end{bmatrix} = \begin{bmatrix}0 & N_i\\ -N_i^T & 0\end{bmatrix}. \tag{35}$$

Every transformation (33) for which all $\alpha + \beta\lambda_i$ are nonzero, converts the summands $(I_{l_i}, J_{l_i}(\lambda_i))^\triangledown$ of (29) to the pairs

$$(\alpha I_{l_i} + \beta J_{l_i}(\lambda_i), \gamma I_{l_i} + \delta J_{l_i}(\lambda_i))^\triangledown\,;$$

by (35) they are congruent to

$$\big(I_{l_i},\ (\alpha I_{l_i} + \beta J_{l_i}(\lambda_i))^{-1}(\gamma I_{l_i} + \delta J_{l_i}(\lambda_i))\big)^\triangledown. \tag{36}$$

The matrices $\alpha I_{l_i} + \beta J_{l_i}(\lambda_i)$ and $\gamma I_{l_i} + \delta J_{l_i}(\lambda_i)$ are triangular; their diagonal entries are $\alpha + \beta\lambda_i$ and $\gamma + \delta\lambda_i$. Hence, the pair (36) is congruent to

$$\left(I_{l_i},\ J_{l_i}\left(\frac{\gamma + \delta\lambda_i}{\alpha + \beta\lambda_i}\right)\right)^\triangledown,$$

and the sequence of eigenvalues changes by the rule (31).

By Lemma 6, the matrices A and B in (28) must be linearly independent. As follows from (29), A and B are linearly dependent only if (30) holds. $\square$

Remark 10. The theory of Lie rings and algebras is tied to the theory of groups; see [1, Section 7] or [4]. In particular, the results of Sections 3 and 4 are easily extended to every p-group G being the semidirect product of the central commutator subgroup G' of type $(p, \dots, p)$ and an abelian group of type $(p, \dots, p)$. If G is such a group, then

$$G' = \langle a_1\rangle_p \times \cdots \times \langle a_t\rangle_p, \qquad G/G' = \langle c_1\rangle_p \times \cdots \times \langle c_n\rangle_p.$$

Choosing $b_i \in c_i$, we may give G by the defining relations

$$a_l^p = b_i^p = 1, \qquad [a_l, a_r] = [a_l, b_i] = 1, \qquad [b_i, b_j] = a_1^{\alpha_{1ij}} \cdots a_t^{\alpha_{tij}},$$

in which $l, r \in \{1, \dots, t\}$, $i, j \in \{1, \dots, n\}$, and

$$A_1 = [\alpha_{1ij}], \dots, A_t = [\alpha_{tij}]$$

are linearly independent skew-symmetric n-by-n matrices over the field $\mathbb{F}_p$ of p elements. Conversely, each tuple $(A_1, \dots, A_t)$ of linearly independent skew-symmetric n-by-n matrices over $\mathbb{F}_p$ gives such a group, and two tuples give isomorphic groups if and only if one reduces to the other by congruence transformations and substitutions (19), in which the matrix $[\gamma_{ij}]$ is nonsingular. Reasoning as in Theorem 9, we can describe such groups having G' of order p^2. (A canonical form for congruence of a pair of skew-symmetric matrices over an arbitrary field is a direct sum of pairs of the form (32) with the Frobenius blocksa instead of the Jordan blocks $J_m(\lambda)$.) The problem of classifying such groups with G' of order p^3 is hopeless since it reduces to the problem of classifying pairs of matrices over $\mathbb{F}_p$ up to similarity. By [9], the problem of classifying finite p-groups with central commutator subgroup of order p^2 is hopeless in the same way both for the groups in which G' is cyclic and for the groups in which G' is of type (p, p). All finite p-groups with central commutator subgroup of order p are easily classified; see [5] and [10].

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